1. **Orbital energetics [20 pts]**. A comet with a mass of $2 \times 10^{14}$ kg orbits the Sun with a semimajor axis of 20 AU and an eccentricity of 0.75.

   (a) **[8 pts]** Calculate the speed (in km s$^{-1}$) at which it is moving at *perihelion* and at *aphelion*.

   (b) **[8 pts]** For both the *perihelion* and at *aphelion* cases, calculate the kinetic energy (in J) that must be lost or gained (specify which!) in order to circularize the comet’s orbit.

   (c) **[4 pts]** In light of your results, explain why, for a given amount of energy transferred during an encounter, it is easier to ‘raise’ than to ‘lower’ a comet’s orbit.
2. **Atmospheric pressure** [15 pts]. The Earth’s atmosphere has a mean molecular weight of 29, an average temperature of 290 K, and an estimated total mass of $5 \times 10^{18}$ kg. Using these values, and approximating the atmosphere as isothermal,

(a) [5 pts] Estimate the surface pressure in N m$^{-2}$.
(b) [5 pts] Estimate the pressure scale height $H$ in km.
(c) [5 pts] Show that, at an altitude of $H$, typical molecular speeds due to thermal motions are still much less than the Earth’s escape speed.
3. **Exoplanet Properties [20 pts]**. The plot below shows a schematic radial velocity curve for a star (with estimated mass $M_\star = 0.5 M_\odot$) being orbited by a single planet.

(a) **[5 pts]** Calculate the semimajor axis of the planet’s orbit in AU. You may assume the planet’s mass is much less than the star’s.

(b) **[8 pts]** Assuming circular orbits, calculate the minimum mass of the planet in units of the Earth’s mass. Explain why this is a minimum mass.

(c) **[7 pts]** Draw a schematic light curve for the system assuming it is viewed edge-on so both primary and secondary eclipses are possible. Label your light curve with the same time steps A, B, and C as in the radial velocity curve.
4. **Ionization balance** [15 pts]. An O star puts out a certain number of ionizing photons, \( Q_* \), every second. Assume enough time has passed so that ionization balance can be assumed.

(a) [5 pts] Within the volume of the H\(\text{\textsc{ii}}\) region around the star, how many recombinations occur every second?

(b) [5 pts] For a fully ionized hydrogen gas \((n_e = n_p)\), write an expression for the radius of the H\(\text{\textsc{ii}}\) region in terms of \( Q_* \), the electron density \( n_e \), and the recombination rate coefficient \( \alpha_B \) (assumed equal to \( 2.6 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1} \) for gas at an electron temperature of \( 10^4 \text{ K} \)).

(c) [5 pts] Derive an expression for the mass of the H\(\text{\textsc{ii}}\) region in terms of \( Q_* \), \( n_e \), and \( \alpha_B \). Evaluate your result for \( Q_* = 10^{48} \text{ photons s}^{-1} \), \( n_e = 10^2 \text{ cm}^{-3} \), and \( T_e = 10^4 \text{ K} \), converting your answer to \( M_{\odot} \).
5. Cooling time [15 pts].

(a) [12 pts] Estimate the length of time (in yr) it would take a neutral hydrogen cloud to cool from 100 K to 50 K by emitting optically thin 21-cm radiation. Assume a number density of 1 cm$^{-3}$ and an energy density of $\epsilon = \frac{3}{2}nkT$.

(b) [3 pts] Briefly comment on whether 21-cm radiation is an important cooling mechanism in the neutral ISM.

For the 21-cm line, the frequency is 1420 MHz, the Einstein $A$ coefficient is $2.87 \times 10^{-15}$ s$^{-1}$, and the upper and lower states have statistical weights of 3 and 1 respectively.
6. **Dust cloud [15 pts]**. A spherical dust cloud of radius 1 pc has a wavelength-dependent absorption coefficient given by

\[ \alpha_\lambda = \alpha_0 \left( \frac{\lambda}{\lambda_0} \right)^{-2}, \]

where \( \alpha_0 \) is the cross section per unit volume (units: m\(^{-1}\)) at some reference wavelength \( \lambda_0 \).

(a) **[7 pts]** Suppose the cloud is estimated to have an optical depth through the center of the cloud of \( \tau_V = 1 \) at a wavelength of \( \lambda = 550 \) nm, corresponding to the \( V \) band. Determine the value of \( \alpha_0 \lambda_0^2 \) for this case.

(b) **[8 pts]** Calculate the ratio of total to selective extinction, \( A_V/E(B-V) \), for this extinction law. You may assume \( \lambda_B = 440 \) nm.